THE BRUUN RULE—THE RELATIONSHIP OF SEA-LEVEL CHANGE TO COASTAL EROSION AND DEPOSITION

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ABSTRACT: The Bruun Rule in shore erosion is presented systematically, starting from initial assumptions and axioms towards a rigorous mathematical treatment. Mathematical treatment is given with detail to assist geologists in full understanding of the development. Cases of beach excavation, littoral drift, artificial beach nourishment and formation of a cuspate shoreline are treated by the theory, based on the Bruun Rule.

INTRODUCTION

The concept of shore erosion due to sea-level rise expressed by Bruun (1962), later becoming widely known as the "Bruun Rule" (Schwartz 1967), belongs to the class of concepts, which are enthusiastically supported by some scientists, while immediately triggering a negative response from others.

The chronology of Bruun Rule studies (Fisher 1980a) includes laboratory and/or field observations, by Schwartz (1965, 1967), Dubois (1975, 1976, 1977a, 1977b), Hands (1976, 1977, 1980), Fisher (1977a, 1977b, 1977c, 1980b), and Rosen (1977, 1978a, 1978b, 1980); all of them strongly supporting Bruun's concepts. On the other hand, the concept was criticized by Swift (1976) as being of limited applicability. Another criticism is that by Kaplin (1973), who is skeptical about the validity of Bruun's concepts; although supportively stating simultaneously that at least some of Bruun's concepts were known by the Soviet school as early as 1946. Valuable comments on applicability of the concept are given by Gill (1979).

It appears that most of the controversy exists solely due to differences in interpretation of the Bruun Rule. The differences arise, in their turn, due to lack of rigorous mathematical formulation as well as lack of clarity in the statements of the initial assumptions of the Rule. Hence, our task in this paper is to present the Bruun Rule as a rigorous theory based on clearly stated assumptions. It is shown that the Bruun Rule has, in fact, a much wider field of application, than was previously thought. The Bruun theory can be applied for both "closed" and "open" beach systems. We start from formulation of the Bruun Rule as in Schwartz (1965, 1967).

THE BRUUN RULE AND ITS INITIAL ASSUMPTIONS

The Bruun Rule states the following (Fig. 1):

1. A rise in sea level causes erosion of the upper beach and shoreward displacement of the shore-water boundary.
2. The change in sea level corresponds to translation of the transverse beach profile while retaining its original shape.
3. The material eroded from the upper beach is equal in volume to the material deposited on the nearshore bottom.
4. The rise of the nearshore bottom is equal to the rise in sea level.
5. The relationship between sea level rise a and shoreward displacement s of the beach profile is given by the formula (Bruun 1962):

$$a = \frac{hs}{l} \tag{1}$$

where: l is the length of the transverse profile,
h is the profile height, being the sum of sea depth at the distance l from the shore and the shore elevation above the sea level.

LIMITS OF THE BRUUN RULE

As stated by Gill (1979) the rule only applies 1, where there is sufficient energy; 2, where equilibrium has been attained; 3, where there is sufficient space in the subtidal area; and 4, if there is sufficient sediment.

EXPERIMENTAL SUPPORT OF THE BRUUN RULE

The two classes of experiments, supporting the Bruun Rule, are laboratory (Schwartz 1965, 1967) and field experiments. The advantage of laboratory experiments is that the "equilibrium profile" which is practically unobservable on a beach, due to continuous variations in wave climate, can be maintained in a laboratory where the wave climate can be set constant.

LABORATORY EXPERIMENTS

In the first experiment (Schwartz 1965), the wave basin was 81.25 cm wide and 115 cm long with variable gradient of the bottom adjustable at 0°, 2.5°, and 4.5°. The waves generated had a period of 0.33 sec ±5%, an amplitude of 8±2 mm and a wave length of 15±1 cm. Water depth ranged from 5 to 10 cm, the sand used was a natural Ottawa sand, washed and sorted.

It was found that 30 mins of wave attack produced a beach profile which did not change with subsequent wave action, and this profile was assumed to be the "original profile" figuring in the Bruun statements. The water level was then raised by 10 mm, the wave

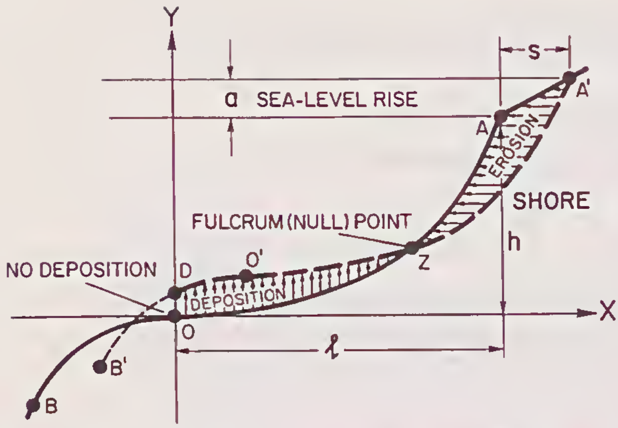

Fig. 1—Definition sketch of the variables in the Bruun Rule. The proto-profile AOB is translated towards the shore (positive x-direction) to its new position $A'O'B'$. However, no deposition occurs at the values of $x<0$. The segment DO is assumed to be the sharp boundary between the regions. In reality there will always be a smooth transition from the point D to the curve BO. However, this is neglected in the Bruun theory.

generator was set up again and as soon as the equilibrium profile was reached water depth and depth of sediment at the outer edge of the shelf were measured.

The profile retained its original shape after water level rise and was translated towards the shore, causing the shore recession. During the experiment the slope of the bottom was varied and it is remarkable that this change did not have any significant effect on water depth or sand heights accumulated at the lower portions of the beach.

In the second laboratory experiment (Schwartz 1967) the wave tank was larger in size (100×232.5 cm) and the wave generator operated with variable periods. The sand was the same as in the previous experiment and water level was raised in different increments and at various rates.

Profile shapes were observed before and after water level rise and the results indicated support for statements 1, 2 and 3 of Bruun's Rule. The most important statement (4), that the increase in sand height deposition on the outer shelf equals the rise of water level (hence, maintaining the constant water depth there), was supported by the experimental results.

These experimental results support the first 4 statements of the Bruun Rule. However, formula (1) was not checked during the laboratory experiments. This was done in some of the following field experiments.

Field experiments

In 1964 an experimental program was established to study the variation of beach profiles under the variation of sea level due to neap and spring tides (Schwartz 1967, 1979). At Herring Cove Beach in the Cape Cod National Seashore (USA) during a slight breeze, small waves approaching the breaker zone were observed with 15-20 cm heights and periods of 3 sec. The direction of wave approach during the summer is predominantly from the north-west. Shore drift at this time, is consequently, toward the south. Sediment supply for this part of the Cape was provided by shore drift from the cliffs of glacial drift that form the outer coast. The nearshore bottom is characterized by a steep slope prior to grading off to a gentler slope.

At nearby Nauset Light Beach which bears 12° west of north and is exposed to the fetch of the Atlantic Ocean, small waves from the southeast approach the breaker zone on a calm day with a period of 8 secs and a height of 30 cm. Erosion of the back beach cliff of glacial drift supplies sediment that drifts northward. The nearshore bottom slopes gently to a bar 500 m offshore at low tide.

The results of profiling during low and high water levels indicate parallel translation without significant change in shape during sea-level rise. The more dynamic regime at Nauset Light Beach means greater translation.

Another field experiment, strongly supporting statement 2 of the Bruun Rule, was conducted by Dubois (1976) who measured nearshore profiles of Lake Michigan as water level seasonally rose from April to July, 1971. There were a total of 17 measurements of beach profiles during this period. The profiles presented by Dubois clearly indicate the parallel translation without significant changes in shape of profiles as water level rose.

Using the Bruun formula (1), in which the value l was taken as a distance from foreshore base to the position of breaking waves, Dubois (1977a) calculated shore recessions and compared them with the observed values for various wave conditions recorded. The agreement between the values of shore recession observed in the field and those calculated on the basis of Bruun's formula (1) was remarkable.

The above field observations were limited to only two sites and a few months in both Schwartz's and Dubois' experiments. The following field experiment by Hands (1976, 1980) was devoted to more long-term shore erosion.

Hands (1976, 1980) described the response of Lake Michigan shores to increased water level over a 9 year period at 34 sites. He used formula (1) and demonstrated good agreement with observed erosion. His most interesting result was the clear transition zone between the area of bottom erosion and no erosion. Hands was the first to point out that actual length of the bottom profile (value l in eq. (1)) is not important, because it is the value h/l, which is critical in the formula (1), and this value does not change much whether the length of a transition zone is considered or not.

Observations by Fisher (1977a, 1977b, 1980b) of Rhode Island shoreline retreat over 35 years with a scope of 113 sites take into account an advance of sea-shore boundary due to a submergence (drowning) of the shore without erosion, separating this from shore retreat due to shore erosion, to which he applies Bruun's formula

(1). The greatest contribution of Fisher is to consider the "open" system, in which not all the sediment volume eroded is deposited on the nearshore bottom. Sediment budget calculations by Fisher require knowledge of the position of the point between the beach erosion zone and the offshore deposition zone (point Z on Fig. 1) to which various names had been prescribed (inflection, fulcrum, null point).

It will be demonstrated below, however, that knowledge of position of such a point is not needed for either "closed" or "open" systems.

The longest project undertaken up to date was that of Rosen (1978b, 1980), who described the shore erosion on 146 beach units, due to sea level rise, over 100 years. The study area consisted of 350 km of estuarine shoreline in the southern half of Chesapeake Bay. Bruun's formula (1) was applied to calculate an average over 100 years' shore recession rate and the results were compared to the measured values.

On the basis of the above laboratory and field experiments it is clear that experimental support exists for the Bruun Rule. We now have to analyze the Bruun Rule statements themselves.

ANALYTICAL DESCRIPTION OF THE BRUUN EFFECT

Statement 2 of the Bruun Rule, termed the "Bruun Effect" (Schwartz 1967), must be accompanied by statement 1, which indicates in what direction (onshore or offshore) the profile is translated during sea level rise. Considering these two statements together as separated for the time being from the other Bruun statements, we must introduce a notation $\bar{a}$ for the rise of the profile, which is not required at this stage to be equal to the sea level rise a.

The bottom profile is considered as a function $f(x,s)$ of the variable x and the parameter s.

Assume that the initial shape of the profile, before erosion occurred ($s=0$), was given by a certain arbitrary function $f_0(x)$. Then the initial condition can be written as

$$f(x,s)\Big|_{s=0} = f_0(x) \tag{2}$$

The translated profile, shifted by s and lifted by $\bar{a}(s)$ is given by

$$f(x,s) = f_0(x-s) + \bar{a}(s) \tag{3}$$

Let us assume now, that we accept Bruun formula (1), equating $a=\bar{a}$, which, when substituted into (3) gives the result:

$$f(x,a) = f_0\left(x - \frac{al}{h}\right) + a \tag{4}$$

This is the analytical form of the Bruun effect, because, when $a=0$, the function $f(x,a)$ reduces to the initial profile shape $f_0(x)$; for any given positive sea level rise a, the profile is shifted shorewards in the positive x-direction by the value al/h and raised by the value a, retaining its original shape $f_0(x)$, hence statements 1 and 2 of Bruun's theory follow from (4). (For the negative a (sea level fall), the profile moves offshore and downwards).

Let us consider now the case in a certain sense opposite to rise in sea level, namely, elevation of the beach profile by the value $b(s)$ due, for example, to artificial nourishment of a beach. Elevation of the beach profile while the absolute sea level remains still is equivalent to a fall of the relative sea level with respect to the nearshore bottom. Hence, according to Bruun's statements 1 and 2, the beach should retain its original shape, but must be translated seawards.

If, once again, the Bruun formula (1) is utilized, with the value of profile rise b substituted instead of a, a formula analogous to (4) emerges:

$$f(x,b) = f_0\left(x + \frac{bl}{h}\right) + b \tag{5}$$

When there is no beach nourishment ($b=0$), the beach profile reduces to the original profile shape $f_0(x)$.

For any positive $b>0$ (beach nourishment) the profile is shifted seawards by the value bl/h, while retaining its original shape. (For b negative due, for example, to sinking of sediments offshore, littoral drift or bottom excavation, the profile is moving shorewards, i.e. erosion occurs.)

If it happens that sea level rise a is accompanied by beach nourishment, resulting in the bottom rise b, then, combining equations (4) and (5), the resultant beach transformation can be described as:

$$f(x,a,b) = f_0\left[x + \frac{l}{h}(b-a)\right] + (b+a) \tag{6}$$

where $f_0(x)$ is the original shape of the beach.

Note that the basic equations (4), (5) and (6) were obtained solely on the basis of the first two statements of Bruun Rule.

BRUUN RULE FOR CLOSED BEACH SYSTEMS

Statement 3 of the Bruun Rule is the definition of the closed beach system, in which the volume of erosion is balanced by the volume of deposition and no exchange of beach material with the outer world exists.

It may seem obvious that for the calculation of volumes of erosion and deposition one needs to know the position of the fulcral (null) point (see Fig. 1) and this need was, in fact, expressed by Dubois (1977a) and Fisher (1980). The difficulty here is that there could be, in principle, several such fulcral points, as shown in Fig. 2. Determination of their positions requires knowledge of beach profile shape and foreknowledge of the same values which one is going to calculate on the basis of equating the volumes of erosion and accretion.

This difficulty, however, can be by-passed if one prescribes opposite signs to the volumes of erosion and accretion according to the signs of difference in ordinates between the protoprofile $f_0(x)$ and the translated profile $f(x,s)$. (Sign plus corresponds to accretion; sign minus to erosion.) Then the definite integral taken along the x-axis will be equal to zero, when volumes (area between the curves in Fig. 2) of erosion and accretion balance each other.

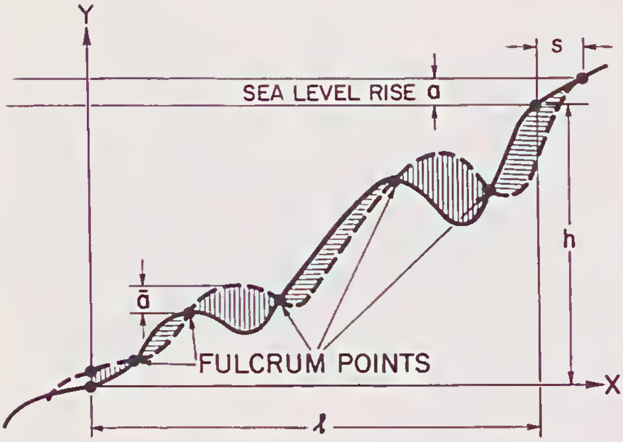

Fig. 2—Multiple fulcrum points, corresponding to multiple areas of erosion (horizontal dashing) and deposition (vertical dashing).

The important issue here is the limits of integration. As already mentioned, the value l (offshore beach length) is supposed to have been chosen in such a way that no erosion or deposition occurs offshore beyond this length, i.e. to the left of the point $x=0$ in Fig. 2. Therefore, it is reasonable to accept point $x=0$ as the lower limit of integration. On the other hand, if the beach profile retreat is equal to value s during sea level rise, it is reasonable to take the upper limit of integration as $(l+s)$. Consequently, we conclude that Bruun's statement 3 can be equivalently written analytically (in the co-ordinate system of Figs 1, 2) as

$$I=\int_0^{l+s}[f(x,s)-f_0(x)]\,dx=0 \tag{7}$$

The first two statements of the Bruun Rule were shown in the previous paragraph to lead to the analytical expression (3) of the function $f(x,s)$, in which the rise of bottom profile as a whole was denoted $\bar{a}(s)$. According to statement 4 we now equate the rise of profile $\bar{a}(s)$ to the sea level rise $a(s)$. Hence, the function $f(x,s)$, figuring in (3) can be written

$$f(x,s)=f_0(x-s)+a(s) \tag{8}$$

Let us emphasize that equations (7) and (8) comprise statements 1-4 of the Bruun Rule, but the Bruun formula (1) (statement 5) is *not* used. On the contrary, we are going to demonstrate, that these four statements form the complete set of axioms, from which formula (1) can be derived. We start from several examples of profiles, having simple analytical expressions.

Analytical Profiles

Parabolic profile

Consider the profile, described by a parabola

$$f_0(x)=x^2 \tag{9}$$

Let us note, that when $x=l$, the right end value of function (9) equals the profile height h (see Fig. 3), hence $h=l^2$. Substituting (9) into (8) and then into (7), one obtains

$$\int_0^{l+s}[(x+s)^2+a(s)-x^2]dx=\int_0^{l+s}[-2xs+s^2+a(s)]dx= \tag{10}$$

$$-x^2s+s^2x+xa(s)\Big|_0^{l+s}=-l^2s-ls^2+(l+s)a(s)=0$$

Hence

$$a(s)=\frac{l^2s+ls^2}{l+s}-\frac{hs(1+\frac{s}{l})}{l(1+\frac{s}{l})}=\frac{hs}{l} \tag{11}$$

where the value l^2 was equated to h. Bruun's formula (1) emerges rigorously from equations (7) and (8) for the parabolic profile.

Now the position of the fulcrum point can be found as the point of intersection of the functions $f_0(x)$ and $f(x,s)$ by solving the equation (notice that $a(s)=hs/l=l^2s/l=ls$)

$$x^2=(x-s)^2+a(s)=(x-s)^2+ls$$

$$0=-2xs+s^2+ls$$

$$x=\frac{s(l+s)}{2s}=\frac{l+s}{2} \tag{12}$$

The fulcrum point appears to be at the distance $(l+s)/2$ from the coordinate system origin.

Let us find the volume (area, as in Fig. 1) of accretion. This is the integral between the points $x=0$ and the fulcrum point $x=½(l+s)$

$$V_A=\int_0^{½(l+s)}[(x-s)^2+ls-x^2]\,dx=s\int_0^{½(l+s)}[(l+s)-2x]\,dx$$

$$=s[(l+s)x-x^2]\Big|_0^{½(l+s)}=¼s(l+s)^2 \tag{13}$$

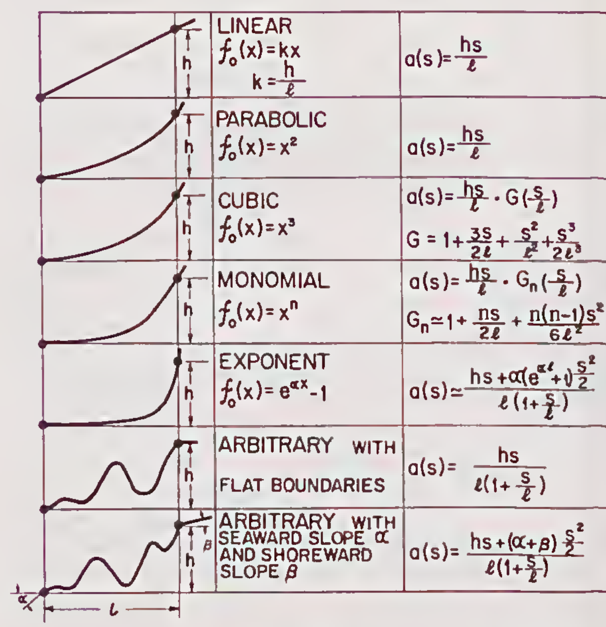

Fig. 3—Analytical and non-analytical profiles with corresponding formula, relating sea level rise a to shore retreat s.

Analogously, the volume of erosion, i.e. between the fulcrum point $x = \frac{1}{2}(l+s)$ and the shoreward limit $x = l+s$, is equal to

$$V_E = \int_{\frac{1}{2}(l+s)}^{l+s} [(x-s)^2 + ls - x^2]dx = s[(l+s)x - x^2]\Big|_{\frac{1}{2}(l+s)}^{l+s}$$

$$= -\tfrac{1}{4}s(l+s)^2 \qquad (14)$$

From this example it is seen that volumes of erosion and accretion are equal in absolute values, as it should be according to the Bruun statement 3.

However, the actual volumes can be found *only* when the position of the fulcral point has been calculated. The impressive result obtained was that the Bruun formula is exact for the parabolic profile. We shall see now, that this is *not* necessarily correct for other profile shapes.

Cubic profile

$$\text{Let } f_0(x) = x^3 \qquad (15)$$

then at the right end of the profile, where $x = l$, the value $f_0(x)|_{x=l} = l^3 = h$, the value h being the profile height in the coordinate system of Fig. 3. Substituting (15) into (8) and then into (7), we arrive at

$$I = \int_0^{l+s} [(x-s)^3 + a(s) - x^3]dx = -x^3s + \frac{3}{2}x^2s^2 - xs^3 + xa(s)\Big|_0^{l+s}$$

$$= l^3s + \frac{3}{2}l^2s^2 + ls^3 + \frac{1}{2}s^4 - (l+s)\,a(s) = 0 \qquad (16)$$

Using the identity $l^3 = h$, it follows:

$$a(s) = \frac{hs\left(1 + \frac{3s}{2l} + \frac{s^2}{l^2} + \frac{1}{2}\frac{s^3}{l^3}\right)}{l\left(1 + \frac{s}{l}\right)} = hs/l.\,G(s/l) \qquad (17)$$

One can recognize here the right-hand side of the Bruun formula, being multiplied by the correction factor $G(s/l)$. When the ratio s/l is small in comparison to unity, the value of the correction factor is also small. For example, when $s/l = 0.1$, the value $G(s/l) = 1.05$, meaning that use of Bruun formula (1) for a cubic profile results in an error of 5%. For the more realistic value $s/l = 0.01$, the factor $G(s/l)$ equals 1.005, meaning an error of 0.5%. We show below, that for other analytical profiles the errors could be much larger.

Monomial profile

$$\text{Let } f_0(x) = x^n \qquad (18)$$

where $n \geq 1$. At the point $x = l$ the height of the profile is $h = l^n$. Substituting (18) into (8) and then (7), we can expand the integrand:

$$(x-s)^n + a(s) - x^n = -nx^{n-1}s + \frac{n(n-1)}{2!}x^n s^2$$

$$-\frac{n(n-1)(n-2)}{3!}x^{n-3}s^3 + \ldots (-1)^n s^n + a(s). \qquad (19)$$

Upon integration, the powers of $(l+s)$ appear, which are expanded in the series such as

$$(l+s)^n = l^n\left(1 + \frac{s}{l}\right)^n = l^n\left[1 + n.\frac{s}{l} + \frac{n(n-1)}{2}.\frac{s^2}{l^2} + \ldots\right]$$

$$(l+s)^{n-1} = \left(1 + \frac{s}{l}\right)^{n-1} = l^{n-1}\left[1 + (n-1)\frac{s}{l} + \frac{(n-1)(n-2)}{2}.\frac{s^2}{l^2}\ldots\right]$$

Preserving only the terms with the value s/l of power two and less, one obtains in the same manner as for the parabola and cubic function

$$a(s) \doteq \frac{l^n s\left[1 + \frac{n}{2}.\frac{s}{l} + \frac{n(n-1)}{6}\frac{s^2}{l^2}\right]}{l(1+s/l)} = hs/l.\,G_n(s/l) \quad (20)$$

It is seen, that for large n the value of the correction factor can also be large and departure from the Bruun formula can be significant. For example, for $n = 10$ and $s/l = 0.1$ the value $G_n = 1.5$. We shall discuss these results later and consider now another class of profiles, for which departure from the Bruun formula can also be large.

Exponential profile

$$\text{Let } f_0(x) = e^{\alpha x} - 1 \qquad (21)$$

The profile has a zero ordinate when $x = 0$ and the height of the profile at $x = l$ is equal to $h = e^{\alpha l} - 1$. By varying the (positive) parameter a various shapes can be obtained (Fig. 3). Substituting the translated function

$$f(x,s) = e^{\alpha(x-s)} - 1 + a(s) = e^{\alpha x}e^{-\alpha s} - 1 + a(s) \qquad (22)$$

together with (21) into the integral (7) we obtain, with some simple manipulations

$$I = \int_0^{l+s} [e^{\alpha x}(e^{-\alpha s} - 1) + a(s)\,dx \qquad (23)$$

$$= \frac{1}{\alpha}(e^{-\alpha s} - 1)\,(e^{\alpha l}e^{\alpha s} - 1) + (l+s)a(s)$$

$$= \frac{1}{\alpha}[e^{\alpha l}(1 - e^{\alpha s}) + (1 - e^{-\alpha s})] + (l+s)a(s) = 0$$

As the value s is considered to be small, we employ now the familiar Taylor expansions, retaining only the terms with s in the power two and less

$$1 - e^{\alpha s} = 1 - \left(1 + \alpha s + \frac{\alpha^2 s^2}{2} + \ldots\right) \simeq -\left(\alpha s + \frac{\alpha^2 s^2}{2}\right)$$

$$1 - e^{-\alpha s} = 1 - \left(1 - \alpha s + \frac{\alpha^2 s^2}{2} - \ldots\right) \simeq \alpha s - \frac{\alpha^2 s^2}{2} \qquad (24)$$

Substitution of (24) into (23) leads to

$$-s(e^{\alpha l} - 1) - \frac{\alpha s^2}{2}(e^{\alpha l} + 1) + (l+s)a(s) = 0$$

wherefrom, if one recalls that $(e^{\alpha l} - 1) = h$, the result emerges

$$a(s) = \frac{hs + \alpha(e^{\alpha l} + 1)s^2/2}{l\left(1 + \frac{s}{l}\right)} \qquad (25)$$

It is seen that for α small in comparison with unity the Bruun formula follows once again. However, for large α (see Fig. 3) the second term in the numerator may become predominant and it grows alarmingly fast with the increase of α. We again temporarily postpone the discussion of the reasons for such large errors and instead return back to the case of the linear profile, which we avoided discussing before, quite deliberately.

Linear profile

Let $f_0(x) = kx;\ f(x,s) = k(x-s) + a(s)$ (26)

where the slope $k = \frac{h}{l}$

The integral (7) then reduces to:

$$I = \int_0^{l+s} [k(x-s) + a(s) - kx]dx = \int_0^{l+s} [-ks + a(s)]dx$$

$$= -ks + a(s) = 0 \tag{27}$$

Hence,

$$a(s) = ks = hs/l \tag{28}$$

and the Bruun formula emerges, exactly. Notice, that in this particular case of linear profile (and in this case only) the integrand in (27) equals zero identically, independently of the value of x, as soon as (28) is used. If we recall the case of the parabolic profile, where to find the fulcrum point we need to equalize the integrand to zero, it becomes obvious that for a linear profile every point x is the fulcrum point; hence no erosion or accretion really occurs.

As any analytical function $f(x)$ can be expanded as a power series, it follows from equations (11), (17), (20) and (28), that Bruun formula (1) is rigorous only for generalised parabolic profile

$$f(x) = c_1x + c_2x^2 \tag{29}$$

where the constants c_1 and c_2 are arbitrary. For higher order of profiles the Bruun formula (1) holds only as an approximation, as summarised in Fig. 3.

Non-Analytical Profiles and Boundary Conditions

One may notice that when the profile $f_0(x)$ is shifted shorewards by the value s, the portion of the profile which originally corresponded to the values $x<0$, appears at the positive part of the x-axis (see Figs 1, 2).

This means, that it is insufficient to know the profile shape $f_0(x)$ strictly within the interval $0 \le x \le l$; to calculate the integral (7) one needs also to know the behavior of the function $f_0(x)$ *beyond* the points 0; l, namely, in the wider interval $[-s; l+s]$.

Hence, the outer boundaries of the segment $[0;l]$ are important, and the shape of the profile at these boundaries $-s \le x \le 0$ and $l \le x \le l+s$ forms some kind of boundary condition (not to be confused with the boundary conditions for differential equations).

We are going to demonstrate now that these are precisely the boundary conditions, which determine validity or invalidity of the Bruun formula (1).

Let us consider an arbitrary function $f(x)$ with its domain of definition in the interval $[-s<x<l+s]$. (The subscript zero is dropped out here for simplicity of notation only.)

Using the general formulae (7) and (8), the integral I is as follows:

$$I = \int_0^{l+s} \{f(x-s) + a(s) - f(x)\}dx = I_1 + I_2 - I_3 \tag{29}$$

where

$$I_1 = \int_0^{l+s} f(x-s)dx;\ I_2 = \int_0^{l+s} a(s)dx;\ I_3 = \int_0^{l+s} f(x)dx \tag{30}$$

By a change of variables the integrals may be rewritten as:

$$I_1 = \int_0^{l+s} f(x-s)dx = \int_{-s}^{l} f(x)dx = \int_0^{l} f(x)dx + \int_{-s}^{0} f(x)dx = \tag{31a}$$

$$= \int_0^{l} f(x)dx + \int_0^{s} f(-x)dx.$$

$$I_2 = \int_0^{l+s} a(s)dx = (l+s)a(s) \tag{31b}$$

$$I_3 = \int_0^{l+s} f(x)dx = \int_0^{l} f(x)dx + \int_l^{l+s} f(x)dx = \int_0^{l} f(x)dx +$$

$$\int_0^{s} f(x+l)dx \tag{31c}$$

Combining the expressions (31a, b and c) according to (29) it is seen, that the integrals $\int_0^l f(x)dx$ cancel each other and:

$$I = \int_0^{s} [f(-x) - f(x+l)]dx + (l+s)a(s) \tag{32}$$

According to equation (7), the value I should equal zero and the rigorous expression for the relationship $a(s)$ follows:

$$a(s) = \frac{\int_0^{s} [f(x+l) - f(-x)]dx}{l+s} \tag{33}$$

Returning back to the temporarily dropped notation $f_0(x)$, it is convenient to rewrite (33) in the following form:

$$a(s) = \frac{F(s)}{l+s};\ F(s) = \int_0^{s} [f_0(x+l) - f_0(-x)]dx \tag{34}$$

The main advantage of (34) is that it makes clear the role of boundary conditions. It is seen from (34) that behavior of the profile $f_0(x)$ *outside* the interval $[0; l]$ contributes to the value F(s), while shape of the profile *within* this interval is of no importance at all.

To clarify this statement, consider the two classes of non-analytical profiles.

(1) Consider one class of profiles, which are *arbitrary* between the points $[0, l]$ and *horizontal* beyond these points (Fig. 3). The profile is assumed to have the height h, then

$$f_0(x+l) \equiv f_0(l) = h$$
$$f_0(-x) \equiv f_0(0) = 0 \tag{35}$$

Calculation of the function F(s) in (34) gives, after substitution of (35) in it

$$F(s) = \int_0^{s} h\,dx = hs \tag{36}$$

Hence, for such a profile

$$a(s) = \frac{hs}{l+s} = \frac{hs}{l(1+\frac{s}{l})} \tag{37}$$

and Bruun formula (1) follows, if the value s/l is small in comparison to unity.

(2) Consider another class of profiles, which again are *arbitrary* between the points $[0, l]$, but have extensions

beyond the terminal points as linear funetions (Fig. 3)

$f_0(x+l) = h + \beta x$ – shoreward

$f_0(-x) = -\alpha x$ – seaward (38)

The parameter α is, in fact, the seaward slope of the profile, while β is the shoreward slope.

Substitution of (38) into (34) gives

$$F(s) = \int_0^s (h + \beta x + \alpha x] dx = [hs + (\beta + \alpha)\frac{s^2}{2}] \quad (39)$$

It is seen, that the seaward slope α is aeeompanied in formula (39) by the corresponding shoreward slope β and *both* slopes play equal roles.

Partieularly, when the slopes are horizontal, (i.e. $\alpha = \beta = 0$) formula (39) reduces to (36).

Another case of reduetion to (36) occurs when $\alpha = -\beta$, i.e. seaward and shoreward slopes are of opposite signs.

An interesting case emerges when slopes $\alpha = \beta = k$, where $k = h/l$. Then it follows from (39)

$$F(s) = hs + ks^2 = hs(1 + \frac{s}{l}) \quad (40)$$

and the Bruun formula appears rigorously once again

$$a(s) = \frac{hs(1 + \frac{s}{l})}{l+s} = \frac{hs(1 + \frac{s}{l})}{l(1 + \frac{s}{l})} = hs/l \quad (41)$$

Henee, when both seaward and shoreward slopes of otherwise *arbitrary* profiles are equal to the value h/l of the profile, the Bruun formula (1) is exaet.

The results obtained so far for the profiles of various shapes are depicted on Fig. 3 with the corresponding exact formula $a(s)$.

One can see from Fig. 3 that steepness of monomial and exponential profiles grows with the inerease of their order and so does the error in the use of the Bruun formula (1). Hence, *the boundary conditions, rather than the shape of the profiles, are responsible for accuracy of the Bruun formula.* It is seen from (39), that the value $F(s)$ ean depart from the Bruun value hs if the seaward and/or shoreward slopes are large. Reversely, one can always estimate the error, given by the Bruun formula, by means of exaet formula (34).

Consequently, we may eonclude that Bruun's formula (1) follows from the exaet formula (34) as a very aecurate approximation, providing the seaward and shoreward slopes of the profile are not too steep. In turn, formula (34) is based rigorously on equations (7) and (8), which are the mathematical formulation of the first four statements of the Bruun rule.

We show that the same Bruun statements form the basis for the theory of erosion/aecretion in open beaeh systems.

BRUUN RULE FOR OPEN BEACH SYSTEMS

It is convenient to define an open beaeh system as one in which an additional volume (+V) of sediment is supplied from outside the system (for example, beaeh nourishment and/or littoral drift) or, alternatively, some volume of sediment (−V) is removed from the system (for example, due to exeavation of the nearshore bottom and/or to littoral drift).

Beach Nourishment

Consider the ease of a positive volume (+V) of sediment being supplied to the system, while the sea level a is still. This case was partly touched in section 4 where equations (3) and (4) were derived solely on the basis of the first two of Bruun's statements.

In this section we denote the horizontal profile displacement due to supply or removal of sediment as r, leaving the notation s for horizontal profile retreat solely due to sea-level rise. Correspondingly, equation (3) in the new notation is

$$f(x,r) = f_0(x+r) + b(r) \quad (42)$$

where $b(r)$ is positive, when rise of the profile occurs due to income of positive volume (+V) of sediment into a beaeh system and horizontal displaeement r due to the same factor is taken as positive for *seaward* displaeement of the profile.

The increment in volume of sediment in a beaeh system due to translation of the profile given by (42) can be ealculated as an integral I, analogous to (29).

However, the limits of integration must be different now. As the profile is shifting seawards (positive r), part of the shifted profile will appear *left* of point $x=0$ (Fig. 4), beyond the limiting length l offshore. But our initial assumption was that a profile is undisturbed to the left of point $x=0$. Henee, the lower limit of integration should be taken as $x=0$.

On the other hand, as the profile is shifted seaward, no disturbance of the profile oecurs beyond point $x=l$, which should be taken now as the upper limit of integration. Consequently, the additional volume (+V) of sediment supplied to the beach should be equal to the integral (we again temporarily drop the subscript zero in $f_0(x)$)

$$I = \int_0^l [f(x+r) + b(r) - f(x)]dx = I_1 + I_2 - I_3 = V \quad (43)$$

where

$$I_1 = \int_0^l f(x+r)dx = \int_r^{l+r} f(x)dx + \int_r^l f(x)dx + \int_l^{l+r} f(x)dx \quad (44a)$$

$$I_2 = \int_0^l b(r)dx = lb(r) \quad (44b)$$

$$I_3 = \int_0^l f(x)dx. \quad (44c)$$

Adding to I_1 eq. (44a) the self-eancelling pair of integrals

$$\int_0^r f(x)dx - \int_0^r f(x)dx \quad (45)$$

we ean rewrite I_1 as

$$I_1 = \int_0^r f(x)dx + \int_r^l f(x)dx + \int_l^{l+r} f(x)dx - \int_0^r f(x)dx \quad (46)$$

where the sum of the first two integrals exactly equals I_3, thus eaneelling it.

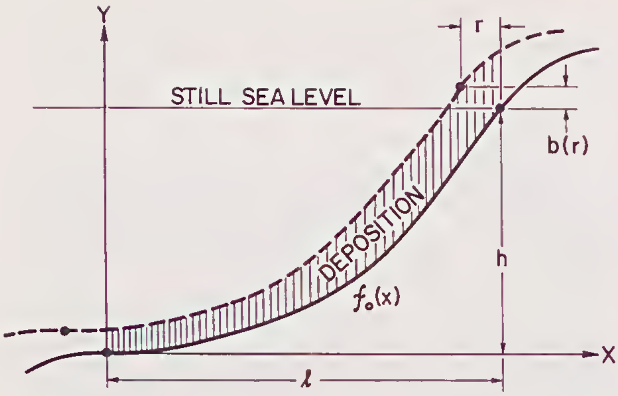

Fig. 4 — Deposition of sediments, resulting in rise of profile by the value $b(r)$ and seaward shift by the value r (r is positive seaward).

Consequently

$$1 = \int_l^{l+r} f(x)dx - \int_0^r f(x)dx + lb(r) = V \tag{47}$$

and the formula for $b(r)$ emerges (returning back to subscript zero)

$$b(r) = \frac{1}{l}\left(V - \int_l^{l+r} f_0(x)dx + \int_0^r f_0(x)dx\right) \tag{48}$$

One has to notice that the profile shape between the points $x=r$ and $x=l$ does not figure in (48) and value of the integrals is determined by the behavior of the profile shape in the neighborhood of its terminal points $x=0$ and $x=l$. Let us assume that the profile is nearly flat around $x=0$ and $x=l$, i.e. at the interval $[0,r]$ the function $f_0(x)=0$ and at the shoreward end $f_0(x)=h$ at the interval $[l, l+r]$, then the first integral in (48) reduces to the value hr and the second to zero. Formula (49) then emerges as:

$$b(r) = (V - hr)/l \tag{49}$$

It can be shown, by analogy with the previous section, that (49) is a very good approximation of the exact result of (48) when the boundary conditions are such that the seaward and shoreward slopes are not too steep and the difference in their slope angles is small.
The expression for r follows from (49)

$$r = (V - bl)/h \tag{50}$$

On the other hand, we did have another expression for the same value r in equation (5)

$$r = \frac{bl}{h} \tag{51}$$

Combining (50) and (51), we obtain

$$r = \frac{V}{2h},\ b = \frac{V}{2l} \tag{52}$$

Consequently, equation (42), becomes dependent on the volume of sediment supplied to the beach

$$f(x, V) = f_0\left(x + \frac{V}{2h}\right) + \frac{V}{2l} \tag{53}$$

We want to emphasize, that the values r and b in (52) and formula (53) are independent of the profile shape.

If the beach loses sediment (due to excavation of the bottom, littoral drift, etc.), then the value V in (52) and (53) should be taken as negative and the beach profile shifts shorewards and downwards, because the values r and b become negative.

When beach nourishment (excavation) is accompanied by sea level rise, it follows from equations (6) and (53), that the new beach profile is described by the equation

$$f(x,a,V) = f_0\left[x + \frac{l}{h}\left(\frac{V}{2} - al\right)\right] + \frac{1}{l}\left(\frac{V}{2} + al\right), \tag{54}$$

where positive values V and a correspond to beach nourishment and rise in sea level.

Equation (54) was obtained, after all, solely on the basis of the first four Bruun Rule statements.

The previous treatment was performed for a two-dimensional cross-section of a beach, which may also be considered a slice (of unit thickness) of an actually three-dimensional beach.

Bruun Rule for Three-Dimensional Beach System with Littoral Drift

Consider the three-dimensional portion of shoreline in the coordinate system of Fig. 5 (note that the y coordinate is now directed alongshore). Any vertical plane, parallel to the plane ZOX, dissecting the shore, forms in its cross-section a two-dimensional profile, such as described in the previous sections. However, there is no need to assume that all the cross-sections have the same or similar shape. For each cross-section with a longshore coordinate y equation (54) holds; therefore independent of the shape of the particular cross-section, its seaward motion (denoted $R(y)$ from now on) and vertical uplift (denoted $\bar{a}(y)$) is equal to

$$R(y) = \frac{1}{h(y)}\left[\frac{V(y)}{2} - al(y)\right] \tag{55a}$$

$$\bar{a}(y) = a + [V(y)/2l(y)] \tag{55b}$$

where a is the rise in sea level as before and the functions $h(y)$; $l(y)$; $V(y)$ are dependent on the longshore coordinate y. Consider the equation of continuity or the volume conservation equation for longshore sediment drift (see, for example, Komar, 1976)

$$\partial V/\partial t = -\partial Q/\partial y$$

where Q is the longshore volumetric rate of sediment transport (measured, for example, in m^3/day).

The physical sense of equation (56) is quite obvious: it states that the time-rate (speed) of volume of sediment removal (sign negative in 56) from a beach equals the longshore rate in change of volumetric transport along the beach. This is illustrated in Fig. 5, where both function $Q(y,t)$ and $\partial Q/\partial y$ are plotted along the longshore coordinate y for one specific type of dependancy $Q(y)$.

It is reasonable to assume further that the basic parameters $h(y)$ and $l(y)$, figuring in (55), do not change much in time due to shore erosion (accretion), and only the values, R, $\bar{a}$, a and V in (55) are variable in time.

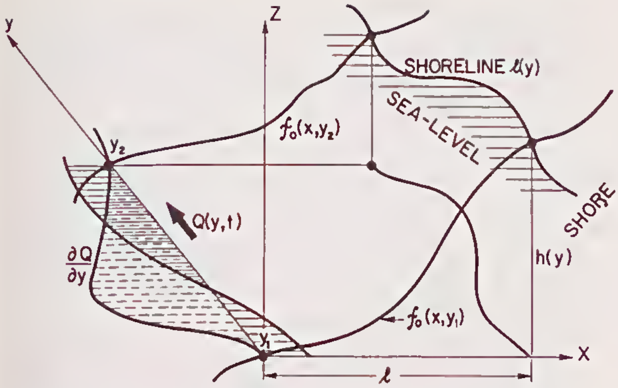

Fig. 5—Definition sketch for longshore littoral drift. Axis y directed alongshore. Profile shapes could be different for various y, such as y_1 and y_2.

Consequently, equations (55) can be rewritten to incorporate the above assumption as

$$R(y,t) = \frac{1}{h(y)}\left[\frac{V(y,t)}{2} - a(t)l(y)\right] \quad (57a)$$

$$\bar{a}(y,t) = a(t) + \frac{V(y,t)}{2l(y)} \quad (57b)$$

Notice that sea level rise a in (57) is assumed to be dependent on time, but independent of longshore coordinate y, meaning that extremely long coastlines, where sea-level varies along the coastline, are not considered. Equations (57) give the complete steady-state solution of the problem of beach nourishment (excavation) in the presence of variations in the time sea level rise $a(t)$ for any distribution of $V(y,t)$ along the shore and in time, providing the values $h(y)$ and $l(y)$ are known.

Differentiating (57) with respect to time t one obtains

$$\partial R/\partial t = ½\partial V/\partial t - l(y)\partial a/\partial t]/h(y) \quad (58a)$$

$$\partial\bar{a}/\partial t = \partial a/\partial t + ½ l(y)\cdot\partial V/\partial t \quad (58b)$$

Substituting instead of $\partial v/\partial t$ in (58) the equal value $-\partial Q/\partial y$ from (56), it follows

$$\partial R/\partial t = -[½\cdot\partial Q/\partial y + l(y)\cdot\partial a/\partial t]/h(y) \quad (59a)$$

$$\partial\bar{a}/\partial t = \partial a/\partial t - ½ l(y)\cdot\partial Q/\partial y \quad (59b)$$

Integrating equations (59) with respect to time, one obtains the final formulae for calculation of shoreward displacement R and profile rise $\bar{a}$ as functions of longshore sediment transport Q and variations in the time of sea level rise $a(t)$

$$R(y,t) = -\frac{l}{h(y)}\left[\frac{l}{2}\int_0^t \frac{\partial Q(y,t)}{\partial y}dt + l(y)\,.\,a(t)\right] \quad (60a)$$

$$\bar{a}(y,t) = a(t) - \frac{l}{2l(y)}\int_0^t \frac{\partial Q(y,t)}{\delta y}dt \quad (60b)$$

We illustrate the use of (60) by two examples.

Example 1

Let $a(t)=0$ and $Q(y,t)=y\sin\omega t$, meaning that the sediment transport rate is linearly distributed along the shoreline and varies periodically in time.

Then

$$\partial Q(y,t)/\partial y = \sin\omega t \quad (61)$$

and from (60) one obtains by integration:

$$R(y,t) = -\frac{1}{2\omega h(y)}\cos\omega t + C \quad (62a)$$

$$\bar{a}(y,t) = -\frac{1}{2\omega l(y)}\cos\omega t + D \quad (62b)$$

where C and D are the constants to be determined from initial conditions. Assuming them as $R(y,t)=0$ and $\bar{a}(y,t)=0$ (no initial erosion—accretion) when $t=0$, it is easy to find from (62)

$$C = \frac{1}{2\omega h(y)};\ D = \frac{1}{2\omega l(y)} \quad (63)$$

hence, the seaward shore displacement and rise of the bottom are:

$$R(y,t) = \frac{1}{2\omega h(y)}(1-\cos\omega t) \quad (64a)$$

$$\bar{a}(y,t) = \frac{1}{2\omega l(y)}(1-\cos\omega t), \quad (64b)$$

meaning periodic (for example; seasonal) variation in shore erosion, *dependent* on the original longshore shape (functions $h(y)$ and $l(y)$) of the coastline.

Example 2

Let $a(t)=0$ and a tropical cyclone (hurricane) moves alongshore, causing a propagating wave-like longshore transport $Q(y,t)$ with magnitude Q_0

$$Q(y,t) = Q_0\cos(Ky+\omega t) \quad (65)$$

Then

$$\frac{\partial Q}{\partial y} = -KQ_0\sin(Ky+\omega t) \quad (66)$$

and upon substitution of (66) into (60a, b) and integration

$$R(y,t) = -\frac{KQ_0}{2\omega h(y)}\cos(Ky+\omega t) + C(y) \quad (67a)$$

$$\bar{a}(y,t) = -\frac{KQ_0}{2\omega l(y)}\cos(Ky+\omega t) + D(y) \quad (67b)$$

Here instead of constants C and D, as in the previous example, the functions $C(y)$ and $D(y)$ should be found from initial conditions. One obtains from (67) for $t=0$

$$C(y) = \frac{KQ_0}{2\omega h(y)}\cos Ky \quad (68a)$$

$$D(y) = \frac{KQ_0}{2\omega l(y)}\cos Ky \quad (68b)$$

and the final result emerges from (67) upon substitution of (68)

$$R(y,t) = \frac{K\,Q_0}{2\omega h(y)}[\cos Ky - \cos(Ky + \omega t)] \quad (69a)$$

$$\bar{u}(y,t) = \frac{K\,Q_0}{2\omega l(y)}[\cos Ky - \cos(Ky + \omega t)] \quad (69b)$$

Note the appearance of the time independent term cos Ky in (69), meaning that after passage of a cyclone (hurricane) an irreversible cuspate change of a coastline occurs, if the rate of sediment transport could be approximated as a propagating wave (65).

DISCUSSION

The first four statements of the Bruun Rule form a non-contradictory set of axioms, which need no other axioms for development of the theory of shore erosion; incorporating not only sea level rise, for which the statements were originally intended, but also the problems of beach nourishment, excavation and littoral drift.

The most important of all assumptions of the theory, its cornerstone, is the statement, that beach profile essentially preserves its shape during process of shore evolution, be it due to sea-level rise, wave attack, beach nourishment etc. Of course, it is recognised, that some temporal variations, perhaps caused by a sudden storm, can be superimposed on the conservative shape of beach profile, but these variations are usually short-lived and can be discarded in the first approximation.

The experimental data available support this assumption. No doubt, further experimental work is necessary.

The most important limitation of the above theory is its static nature.

This means, that the beach system is considered only in two states: initial state, before evolution started and final state, after evolution was completed. The transition process between these two states is beyond the scope of the present theory. Clearly, this shortcoming must be rectified. This is our direction for future research.

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